Journal of Engineering Physics and Thermophysics, Vol. 72, No. 1, 1999

TEMPERATURE FIELD OF A TWO-LAYER CYLINDER WITH VOLUMETRIC HEAT SOURCES AND NONSTATIONARY BOUNDARIES

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UDC 536.21:669.041

A numerical-analytical solution of a differential heat-conduction equation is suggested. Analytical expressions relating the initial and boundary conditions of heat transfer to parabolic temperature distributions in the layers of the cylinder are obtained for the end of the calculated interval of time. The calculation of the temperature field is reduced to the solution of a system of two ordinary differential equations. The error of the solution is analyzed.

Systems of dialog simulation and optimization of thermal technological processes in which multiple calculations of temperature fields are performed specify rather strict requirements on the time of solution of the simulation problem. To reduce the time of calculation, numerical-analytical models in which part of the problem is solved by analytical methods are widely used. This approach allows one to accelerate the process of calculation due to some reduction in the universality of the numerical method.

In what follows, a method of calculation of the temperature field of a two-layer infinite cylinder with variable dimensions in the presence of volumetric heat sources is suggested.

Using the scheme presented in Fig. 1 we can describe: a) a metallic ingot of radius R_1 with a scale layer $R_2 - R_1$; b) a thermally massive cylinder artificially divided into two layers; c) a cylindrical body in which chemical reactions or phase conversions, e.g., moisture evaporation, take place.

An analytical solution relating the initial and boundary conditions of the temperature field to the temperature distribution across the cylinder at the end of the computational time interval $\Delta\tau$ is obtained for the scheme of heat transfer (Fig. 1). Due to this, calculation of the differential equation of heat conduction is replaced by solution of a system of ordinary differential equations.

We assume that the temperature distribution in the cylinder layers at the end of the time interval of computation $\Delta\tau$ is described by the parabolas

$$T_1(X_1) = a_0 + a_2 X_1^2, \quad X_1 = r/R_1, \quad 0 \le X_1 \le 1, \tag{1}$$

$$T_2(X_2) = b_0 + b_1 X_2 + b_2 X_2^2, \quad X_2 = (r - R_1)/(R_2 - R_1), \quad 0 \le X_2 \le 1. \tag{2}$$

The initial conditions are assigned by the mass-mean temperatures of the layers $T_{m,1,b}$, $T_{m,2,b}$. The boundary conditions for the temperature fields with account for relations (1) and (2) are written in the form

$$\frac{\lambda_1 \delta T_1}{R_1 \delta X_1} = 0, \quad r = 0, \quad X_1 = 0; \tag{3}$$

$$q_1 = \frac{2\lambda_1}{R_1} a_2 = \frac{\lambda_2}{R_2 - R_1} b_1, \quad r = R_1, \quad X_1 = 1, \quad X_2 = 0; \tag{4}$$

Ivanovo State Power Engineering University, Russia. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 72, No. 1, pp. 76-79, January-February, 1999. Original article submitted August 22, 1997.

1062-0125/99/7201-0079$22.00

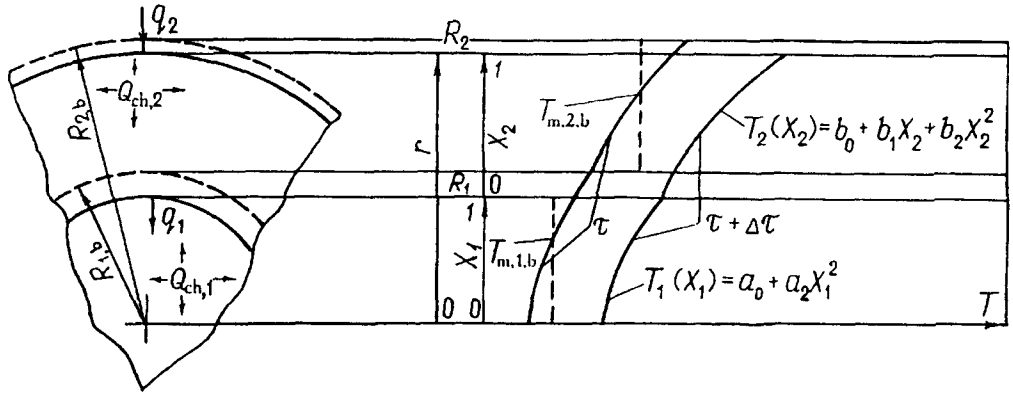

Fig. 1. Calculational scheme of heat transfer in a two-layer cylinder.

$$T_1 (X_1 = 1) = T_2 (X_2 = 0) \quad \text{or} \quad a_0 + a_2 = b_0 , \quad r = R_1 ; \tag{5}$$

$$q_2 = \frac{\lambda_2}{R_2 - R_1} (b_1 + 2b_2) , \quad X_2 = 1 , \quad r = R_2 , \tag{6}$$

where λ_1, λ_2 are the coefficients of thermal conductivity of the layers ($\lambda_1 = \lambda_1(T)$, $\lambda_2 = \lambda_2(T)$); q_1, q_2 are the heat fluxes (see Fig. 1).

The dynamics of the variation of the temperature fields is described by differential equations of heat balance presented in the form of finite differences:

$$\frac{dT_{m,1}}{d\tau} \approx \frac{T_{m,1} - T_{m,1,b}}{\Delta\tau} = \frac{2q_{m,1}}{C_1 R_{m,1}} + \frac{Q_{ch,1}}{C_1 \pi R_{m,1}^2} = \frac{4\lambda_1 a_2}{C_1 R_{m,1} R_1} + \frac{Q_{ch,1}}{C_1 \pi R_{m,1}^2} , \tag{7}$$

$$\frac{dT_{m,2}}{d\tau} \approx \frac{T_{m,2} - T_{m,2,b}}{\Delta\tau} = \frac{2 (q_{m,2} R_{m,2} - q_{m,1} R_{m,1}) + Q_{ch,2}/\pi}{C_2 (R_{m,2}^2 - R_{m,.1}^2)} =$$

$$= \frac{2 (q_{m,2} R_{m,2} - 2\lambda_1 R_{m,1} a_2 / R_1) - Q_{ch,2}/\pi}{C_2 (R_{m,2}^2 - R_{m,1}^2)} , \tag{8}$$

where $q_{m,1}$, $q_{m,2}$, $R_{m,1}$, $R_{m,2}$ are the mean heat fluxes and radii of the surfaces for the time interval $\Delta\tau$:

$$q_{m,1} = (q_1 + q_{1,b})/2 , \quad q_{m,2} = (q_2 + q_{2,b})/2 ; \tag{9}$$

$$R_{m,1} = (R_1 + R_{1,b})/2 , \quad R_{m,2} = (R_2 + R_{2,b})/2 ; \tag{10}$$

$C_1 = C_1(T)$, $C_2 = C_2(T)$ are the mean volumetric heat capacities for the time interval $\Delta\tau$ (J/(m^3·K)) as functions of the temperatures of layers; $Q_{ch,1}$, $Q_{ch,2}$ are the powers of the heat releases from the chemical reactions or phase conversions (W); $T_{m,1,b}$, $T_{m,2,b}$, $T_{m,1}$, $T_{m,2}$ are the mass-mean temperatures at the beginning (b) and end of the computational interval:

$$T_{m,1} = \frac{1}{V_1} \int_{V_1} T (V) \, dV = a_0 + a_2/2 ; \tag{11}$$

$$T_{m,2} = \frac{1}{V_2}\int_{V_2} T(V)\,dV = \frac{1}{\pi(R_2^2 - R_1^2)}\int_0^1 (b_0 + b_1X + b_2X^2) \times$$

$$\times (R_1 + X(R_2 - R_1))(R_2 - R_1)\,dX = b_0 + b_1S_1 + b_2S_2; \tag{12}$$

$$S_1 = (R_1 + 2R_2)/3(R_1 + R_2), \quad S_2 = (R_1 + 3R_2)/6(R_1 + R_2).$$

For known initial and boundary conditions or ones taken as a first approximation the problem of calculation of the temperature field within $\Delta\tau$ is reduced to solution of a system of five linear equations in the coefficients a_0, a_2, b_0, b_1, b_2 of the approximation of the temperature distribution across the cylinder. Compared to the known method of finite differences, even at this level of solution of the problem we can obtain a certain effect due to a reduction in the order of the system of equations. A special choice of the system of spatial coordinates and the form of the integrand allowed one to find an analytical solution for calculation of the heat transfer within the interval $\Delta\tau$.

We consider the order of solution of the system of equations (4)-(8) with account for (1), (2), (9)-(12). We express the coefficient b_1 from Eq. (4):

$$b_1 = \frac{2\lambda_1}{R_1}\frac{R_2 - R_1}{\lambda_2}a_2. \tag{13}$$

We transform (6) accounting for (13):

$$b_2 = \frac{q_2(R_2 - R_1)}{2\lambda_2} - \frac{\lambda_1}{\lambda_2}\frac{R_2 - R_1}{R_1}a_2. \tag{14}$$

We find the coefficient a_0 from Eq. (7) with account for (13) and (14):

$$\frac{a_0 + a_2/2 - T_{m,1,b}}{\Delta\tau} = \frac{4\lambda_1 a_2}{C_1R_{m,1}R_1} + \frac{Q_{ch.1}}{C_1\pi R_{m,1}^2},$$

transformation of which yields

$$a_0 = T_{m,1,b} + \frac{\Delta\tau Q_{ch,1}}{C_1\pi R_{m,1}^2} + \left(\frac{4\Delta\tau\lambda_1}{C_1R_{m,1}R_1} - \frac{1}{2}\right)a_2. \tag{15}$$

We express the coefficient b_0 from Eqs. (5) and (15):

$$b_0 = T_{m,1,b} + \frac{\Delta\tau Q_{ch,1}}{C_1\pi R_{m,1}^2} + \left(\frac{4\Delta\tau\lambda_1}{C_1R_{m,.1}R_1} + \frac{1}{2}\right)a_2. \tag{16}$$

We substitute relation (12) in Eq. (8) with account for (13), (14), (16) and, having transformed it, we obtain an expression for calculation of the coefficient a_2:

$$a_2 = \left[\frac{2q_{m,2}R_{m,2}\Delta\tau + Q_{ch,2}\Delta\tau/\pi}{C_2(R_{m,2}^2 - R_{m,1}^2)} - T_{m,1,b} + T_{m,2,b} - \frac{Q_{ch,1}\Delta\tau}{C_1\pi R_{m,1}^2} - \frac{q_2(R_2 - R_1)}{2\lambda_2}S_2\right] \times$$

$$\times \left[4\lambda_1\Delta\tau\left(\frac{1}{C_2(R_{m,2}^2 - R_{m,1}^2)} + \frac{1}{C_1R_{m,1}R_1}\right) + \frac{1}{2} + \frac{\lambda_1(R_2 - R_1)}{\lambda_2R_1}(2S_1 - S_2)\right]^{-1} \tag{17}$$

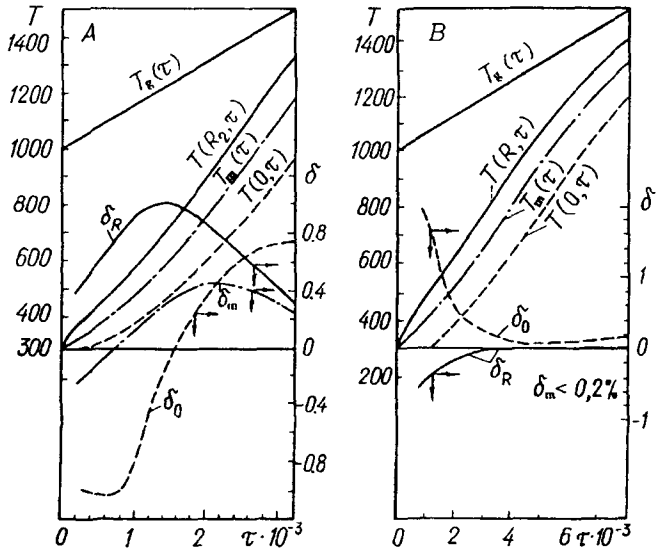

Fig. 2. Temperatures and errors of temperature calculation (δ) for a cylinder with radius $R = 0.2$ m (A) ($R_1 = 0.1$, $R_2 = R = 0.2$) and $R = 0.3$ m (B) ($R_1 = 0.15$, $R_2 = R = 0.3$) in radiative heating. T, K; δ, %; τ, sec.

Starting from calculation of a_2, we can determine the remaining coefficients of the parabolic temperature distribution by formulas (13)-(16) and then calculate any temperature in the cylinder cross section by (1) and (2).

The mass-mean temperature of the entire cylinder is determined by the formula

$$T_m = [T_{m,1}R_1^2 + T_{m,2}(R_2^2 - R_1^2)]/R_2^2. \tag{18}$$

The dynamics of the temperature-field variation can be calculated by existing algorithms for solution of ordinary differential equations by a numerical method. As applied to problems of this type these algorithms are considered in [1, 2]. In [2] a technique for calculation of the thermal effect of chemical reactions and the dynamics of dimension variation caused by transformation of substances (e.g., steel oxidation) is described.

To test the solutions obtained and evaluate the error of determination of the parameters of the temperature field and the applicability range of the method, we calculated the radiative heating of a cylinder ($\sigma = 4.5 \cdot 10^{-8}$ W/(m$^2 \cdot$K^4) is the coefficient of radiative heat transfer, $T_g(\tau) = 1000 + 0.0625\tau$ is the gas temperature) with variable thermophysical properties ($\lambda(T) = 40 - 0.01T$, $C(T) = 10^6(40 - 0.01T)/(8 \cdot 0.002T)$). The material of the cylinder layers was taken to be the same.

The errors of calculation of the temperature of the surface $r = R_2$ (δ_R) and the thermal center $r = 0$ (δ_0) and the mass-mean temperature of the entile cylinder δ_m were determined by comparing the results of calculation by the described method with the test temperature field found by the known method of finite differences. Changes in the dimensions of the layers and the heat release were disregarded.

Figure 2 shows the temperature fields of a cylinder with a radius $R = 0.2$ m ($R_1 = 0.1$, $R_2 = 0.2$) and a cylinder with a radius $R = 0.3$ m ($R_1 = 0.15$, $R_2 = 0.3$); in the lower parts of the figure the errors of temperature calculation are given. As follows from the figure, the errors of the calculation for a cylinder with a radius $R = 0.2$ m do not exceed 0.5%, and for a cylinder with a radius $R = 0.3$ m 0.8%.

The error of calculation of the mass-mean temperature, determining the main technical and economic characteristics of the process of heat treatment of materials, was 0.1% for $R = 0.2$ m and 0.2% for $R = 0.3$ m.

On the basis of an analysis of the results of calculation (other initial data included) we can regard the applicability range of the method to be limited by the Stark ($Sk = \sigma T_g^3 R/\lambda < 1$) and Biot ($Bi = \alpha R/\lambda < 2$, where α is the coefficient of convective heat transfer) numbers.

Calculations performed with different values of $\Delta\tau$ showed that the optimum value of $\Delta\tau$ (at this value the error of temperature-field calculation is minimum) can be estimated by the Fourier number $\Delta Fo = \alpha\Delta\tau/R^2 = \lambda\Delta\tau/(CR^2)$, which should be within the range $0.02 < \Delta Fo \leq 0.05$.

Use of the considered method of temperature-field calculation in modeling heating of materials allowed one to reduce considerably the labor consumption of the calculations and employ a more convenient dialog mode of personal-computer operation.

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